Ocular Wavefront Error Representation (ANSI Standard)

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Reasons for Standardization

• Zernike polynomials have traditionally been used to represent wavefronts in optical systems with circular pupils.
• First developed by Fritz Zernike in the 1930s for phase contrast microscopy. Multiple ways of scaling, orienting and ordering the basic polynomials have evolved, which can lead to confusion.
• The standard helps to remove ambiguity to facilitate clarity and exchange between all users.
• Lessons learned from Corneal Topography standards in the 1990s.
References


• ANSI Z80.28-2004 Methods for Reporting Optical Aberrations of Eyes.

• ISO/DIS 24157 (Draft currently being voted on 2007)
Wavefront Error Measurement
Coordinate System

Polar Coordinates:

\[ r = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \]

r ranges from \([0, r_{\text{max}}]\)
\(\theta\) ranges from \([0^\circ, 360^\circ]\)
Normalized Coordinate System

\[ \rho = \frac{r}{r_{\text{max}}} \]

\[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \]

\( \rho \) ranges from \([0, 1]\)

\( \theta \) ranges from \([0^\circ, 360^\circ]\)

ANSI STANDARD

Normalized Polar Coordinates
Another Coordinate System

NON-STANDARD

Normalized Polar Coordinates:

\[ \rho = \frac{r}{r_{\text{max}}} \]

\[ \phi = \tan^{-1}\left(\frac{x}{y}\right) \]

\( \rho \) ranges from \([0, 1]\)

\( \theta \) ranges from \([-180^\circ, 180^\circ]\)
The Third Dimension

The z-axis is perpendicular to the x and y-axes and coincides with the Line of Sight.

The Line of Sight connects the point of fixation to the center of the entrance pupil, which in turn connects the center of the exit pupil to the fovea.
Zernike Polynomials

Radial Polynomial, $\rho$

Azimuthal Frequency, $\theta$

$Z_{0}^{0}$

$Z_{1}^{-1}$

$Z_{1}^{1}$

$Z_{2}^{-2}$

$Z_{2}^{0}$

$Z_{3}^{-3}$

$Z_{3}^{-1}$

$Z_{3}^{1}$

$Z_{4}^{-4}$

$Z_{4}^{-2}$

$Z_{4}^{0}$

$Z_{4}^{2}$

$Z_{4}^{4}$
In general, wavefronts can take on a complex shape and we need a way of breaking down this complexity into more simpler components.
Zernike Decomposition

\[-0.003 \times + 0.002 \times + 0.001 \times\]

Spherical Refractive Error
Cylindrical Refractive Error
Coma
Wavefront Error Representation

- The wavefront error can be represent by

\[ W(\rho, \theta) = \sum_{n,m} a_{n,m} Z_n^m(\rho, \theta) \]

\[ n = 0, 1, 2 \ldots ; m = -n, -n + 2 \ldots n-2, n \]

\[ Z_0^0(\rho, \theta) = 1 \]
\[ Z_1^1(\rho, \theta) = 2\rho \cos \theta \]
\[ Z_1^{-1}(\rho, \theta) = 2\rho \sin \theta \]
\[ Z_2^0(\rho, \theta) = \sqrt{3}(2\rho^2 - 1) \]
\[ Z_2^2(\rho, \theta) = \sqrt{6}(\rho^2 \cos 2\theta) \]
\[ Z_2^{-2}(\rho, \theta) = \sqrt{6}(\rho^2 \sin 2\theta) \]
\[ Z_3^1(\rho, \theta) = \sqrt{8}(3\rho^3 \cos \theta - 2\rho \cos \theta) \]
\[ Z_3^{-1}(\rho, \theta) = \sqrt{8}(3\rho^3 \sin \theta - 2\rho \sin \theta) \]
\[ Z_4^0(\rho, \theta) = \sqrt{5}(6\rho^4 - 6\rho^2 + 1) \]
Generalized Zernike Polynomials

$$Z_m^n (\rho, \theta) = \begin{cases} N_n^m R_n^{|m|} (\rho) \cos m\theta &; \text{for } m \geq 0 \\ -N_n^m R_n^{|m|} (\rho) \sin m\theta &; \text{for } m < 0 \end{cases}$$

Double Index
- $n$ is radial order
- $m$ is azimuthal frequency

Normalization

Radial Component

Azimuthal Component
ANSI Standard Zernikes

constant that depends on n & m

\[ N_n^m = \sqrt{\frac{2(n + 1)}{1 + \delta_{m0}}} \]

Kronecker delta function

\[ \delta_{ab} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases} \]

e.g. \[ N_3^{-3} = \sqrt{\frac{2(3+1)}{1+0}} = \sqrt{8} \]
ANSI Standard Zernikes

\[ R_n^{\pm m}(\rho) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s! [0.5(n + |m|) - s]! [0.5(n - |m|) - s]!} \rho^{n-2s} \]

only depends on \(|m|\) (i.e. same for both sine & cosine terms)

Constant that depends on \(n\) and \(m\)

Powers of \(\rho\)

Factorial Function - \(a! = a (a-1) (a-2) \ldots (3) (2) (1)\)

e.g. \[ R_3^{-3}(\rho) = \sum_{s=0}^{(3-3)/2} \frac{(-1)^s (3-s)!}{s! [3-s]! [0-s]!} \rho^{3-2s} = \frac{3!}{3!} \rho^3 = \rho^3 \]
Example Zernike Polynomial

\[ Z_3^{-3}(\rho, \theta) = \begin{cases} 
  N_n^m R_n^m(\rho) \cos m\theta & ; \text{for } m \geq 0 \\
  -N_3^{-3} R_3^{-3}(\rho) \sin(-3\theta) & ; \text{for } m < 0 
\end{cases} \]

\[ Z_3^{-3}(\rho, \theta) = \sqrt{8}\rho^3 \sin(3\theta) \]

Oblique Trefoil
Zernike Polynomials - Single Index

\[ Z_j \quad \text{where} \quad j = \frac{n(n+2)+m}{2} \]

Radial Polynomial, \( \rho \)

Azimuthal Frequency, \( \theta \)

ANSI STANDARD
- Starts at 0
- Left-to-Right
- Top-to-Bottom
Other Single Index Schemes

NON-STANDARD

Starts at 1

cosines are even terms
sines are odd terms


Also Zemax “Standard Zernike Coefficients”
Other Single Index Schemes

NON-STANDARD

Starts at 1
increases along diagonal
cosine terms first
35 terms plus two extra
spherical aberration terms.
No Normalization!!!

Zemax “Zernike Fringe Coefficients”

Also, Air Force or University of Arizona
Other Single Index Schemes

- Born & Wolf
- Malacara
- Others??? Plus mixtures of non-normalized, coordinate systems.

Use two indices $n, m$ to unambiguously define polynomials. Use a single standard index if needed to avoid confusion.
Example 1:

0.25 D of myopia for a 4 mm pupil \( (r_{\text{max}} = 2 \text{ mm}) \)

\[
W = \frac{r^2}{8000} = \frac{(2\rho)^2}{8000} = \frac{\rho^2}{2000} = \frac{1}{4000} Z_0^0(\rho, \theta) + \frac{1}{4000\sqrt{3}} Z_2^0(\rho, \theta)
\]
Examples

Example 2:

1.00 D of myopia for a 2 mm pupil ($r_{\text{max}} = 1 \text{ mm}$)

$$W = \frac{r^2}{2000} = \frac{\rho^2}{2000} = \frac{1}{4000} Z_0^0(\rho, \theta) + \frac{1}{4000\sqrt{3}} Z_2^0(\rho, \theta)$$

Same Zernike Expansion as Example 1, but different $r_{\text{max}}$.

Always need to give pupil size with Zernike coefficients!!
RMS Wavefront Error

- RMS Wavefront Error is defined as

\[
\text{RMS}_{\text{WFE}} = \sqrt{\frac{\iint (W(\rho, \phi))^2 \rho d\rho d\phi}{\iint \rho d\rho d\phi}} = \sqrt{\sum_{n>1, \text{all } m} (a_{n,m})^2}
\]
Orders of Zernike Polynomials

Radial Polynomial, $\rho$

Low Order

Higher Order

Azimuthal Frequency, $\theta$
Single Index Example

Pupil Diameter = 5.5 mm

Zernike Term

Expansion Coefficient (microns)
Mirror Image

- Tendency for high correlation between mirror images of left and right eyes.
- To create mirror image about the y axis: If $n$ is even, negate the coefficients where $m < 0$. If $n$ is odd negate the coefficients where $m > 0$.
- Use for population studies where both eyes are included.

Lombardo et al. JOSA A 23; 777 (2006)
Summary

• Use the two-index scheme $n, m$ to unambiguously define the Zernike polynomials whenever possible.
• If needed, use the standard single index top-to-bottom, left-to-right notation.
• Always include the size of the pupil with the Zernike coefficients.
• Be aware that other schemes exist in the literature and in other fields of optics.