Modal Reconstruction Methods
Pros and Cons

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Introduction

• **Elevation Data**
  – Cornea Topography (usually)
  – Profilometry
  – Interferometry

• **Slope Data**
  – Shack-Hartmann
  – Tscherning
  – Retinal Raytracing
  – Spatially-resolved refractometry
Applications

• Wavefront sensing.
• Ablation pattern design and analysis.
• Dynamic analysis of aberrations and accommodation.
• Tracking biomechanical changes and healing effects with time.
• Feature detection (e.g. keratoconus)
• Optical design - PALs and Custom Contacts
Curve Fitting
Low-order Polynomial Fit

\[ y = 9.9146x + 2.3839 \]

\[ R^2 = 0.9383 \]
High-order Polynomial Fit
Splines
Surface Fitting

- Extension of curve fitting to two dimensions.
- Fitting functions are now a linear combination of fundamental surfaces.
- Splines can also be extended to two dimensions.

\[-0.003 \times + 0.002 \times + 0.001 \times\]
Orthogonal Polynomials

• Complete Sets of Orthogonal polynomials are sets of surfaces which have some nice mathematical properties for surface fitting.
• Examples are Zernike polynomials and Fourier series.
• Taylor polynomials (i.e. 1, x, y, x^2, xy, y^2,....) are not orthogonal.

\[ \int_{A} V_i(x, y)V_j(x, y)\,dx\,dy = \begin{cases} \text{Constant} & i = j \\ 0 & \text{Otherwise} \end{cases} \]
Orthogonality

\[ W(x, y) = \sum_{i} a_i V_i(x, y) \]

\[ W(x, y)V_j(x, y) = \sum_{i} a_i V_i(x, y)V_j(x, y) \]

\[ \int_{A} W(x, y)V_j(x, y)\,dxdy = \sum_{i} a_i \int_{A} V_i(x, y)V_j(x, y)\,dxdy \]

\[ a_j = \int_{A} W(x, y)V_j(x, y)dxdy \cdot \int_{K} W(x_k, y_k)\,V_j(x_k, y_k) \]

A is a circle of unit radius for Zernike polynomials.
A is a square for Fourier series.
• The long part of calculating Zernike polynomials is calculating factorial functions.

\[ Z_n^m (\theta, \phi) = \begin{cases} \quad N_n^m R_n^{|m|} (\theta) \cos m\phi & ; \text{for } m \geq 0 \\ \quad N_n^m R_n^{|m|} (\theta) \sin m\phi & ; \text{for } m < 0 \end{cases} \]

\[ R_n^{|m|} (\theta) = \sum_{s=0}^{(n+|m|)/2} \frac{(\sqrt{1})^s (n \cdot s)!}{s! [0.5(n + |m|) \cdot s] [0.5(n \cdot |m|) \cdot s]} n^{s2} \]
Speed

- Chong et al.* developed a recurrence relationship that avoids the need for calculating the factorials.
- The results give a blazing fast algorithm for calculating Zernike expansion coefficients using orthogonality.

\[
R_{p(q-4)}(r) = H_1 R_{pq}(r) + \left( H_2 + \frac{H_3}{r^2} \right) R_{p(q-2)}(r),
\]

where the coefficients \(H_1, \ H_2\) and \(H_3\) are given by

\[
H_1 = \frac{q(q-1)}{2} - qH_2 + \frac{H_3(p+q+2)(p-q)}{8},
\]

\[
H_2 = \frac{H_3(p+q)(p-q+2)}{4(q-1)} + (q-2),
\]

\[
H_3 = \frac{-4(q-2)(q-3)}{(p+q-2)(p-q+4)}.
\]

*Pattern Recognition, 36;731-742 (2003)*
Least Squares Fit

Conceptually easy to understand, although this can be relatively slow for high order fits.

\[ Z A = F \]

\[ Z^T Z A = Z^T F \]

\[ A = (Z^T Z)^{01} Z^T F \]
Gram-Schmidt Orthogonalization

- Examines set of discrete data and creates a series of functions which are orthogonal over the data set.
- Orthogonality is used to calculate expansion coefficients.
- These surfaces can then be converted to a standard set of surfaces such as Zernike polynomials.

**Advantages**
- Numerically stable, especially for low sampling density.

**Disadvantages**
- Can be slow for high-order fits
- Orthogonal functions depend upon data set, so a new set needs to be calculated for every fit.
Elevation Fit Comparison

32 Orders or 560 total polynomials

0.125 Seconds
Chong Algorithm

10 Seconds
Gram-Schmidt
Keratoconus Detection

Keratoconics

Normals
Keratoconus Detection

Keratoconus Feature

Normal Feature
Keratoconus Detection

\( \hat{K} = \) unit vector of average higher order coefficients of Keratoconus Patients

\( \hat{N} = \) unit vector of average higher order coefficients of Normal Patients

\( \hat{K} = \langle a_{3,3}, a_{3,1}, a_{3,3}, a_{4,4}, \ldots \rangle \)

The dot product of the feature vectors and the vector under test gives a measure of how much the test vector looks like the feature vector.
Keratoconus Detection

35/40 (87%) Cones Correctly Classified
40/40 (100%) Normals Correctly Classified
Misclassified Cones
Progressive Addition Lenses

Progressive Lens manufacturers minimize the details of their designs for proprietary reasons. Typically, PAL patents given a 17 x 17 grid of elevation data.

Spherical Power

Astigmatism

US Patent #6,808,263
Orthogonality

- Slope data is a little trickier since the derivatives of Zernike polynomials are not orthogonal.
- Gavrielides* developed a set of functions that are orthogonal to the gradient of the Zernike polynomials, so expansion coefficients can be calculated as follows:

\[
\begin{align*}
\frac{\partial}{\partial x} W(x, y) &= \sum a_i \frac{\partial}{\partial x} V_i(x, y) \\
\frac{\partial}{\partial y} W(x, y) &= \sum a_i \frac{\partial}{\partial y} V_i(x, y) \\
\frac{\partial}{\partial x} W(x, y) \cdot G_j(x, y) &= \sum a_i \frac{\partial}{\partial x} V_i(x, y) \cdot G_j(x, y) \\
\frac{\partial}{\partial y} W(x, y) \cdot G_j(x, y) &= \sum a_i \frac{\partial}{\partial y} V_i(x, y) \cdot G_j(x, y) \\
\int_a^b W(x, y) \cdot G_j(x, y) dx \ dy &= \sum a_i \int_a^b V_i(x, y) \cdot G_j(x, y) dx \ dy \\
A &\int_a^b W(x, y) \cdot G_j(x, y) dx \ dy \cdot \int_a^b W(x_k, y_k) \cdot G_j(x_k, y_k) dx_k \ dy_k \\
a_j &= \sum a_i \int_a^b V_i(x, y) \cdot G_j(x, y) dx \ dy \cdot \int_a^b W(x_k, y_k) \cdot G_j(x_k, y_k) dx_k \ dy_k
\end{align*}
\]

*Optics Letters 7;526-528 (1982).
Least Squares Fit

\[ Z = \begin{bmatrix} \frac{dV_1(x_1, y_1)}{dx} & \frac{dV_2(x_1, y_1)}{dx} & L & \frac{dV_J(x_1, y_1)}{dx} \\ \frac{dV_1(x_2, y_2)}{dx} & \frac{dV_2(x_2, y_2)}{dx} & L & \frac{dV_J(x_2, y_2)}{dx} \\ \frac{dV_1(x_N, y_N)}{dx} & \frac{dV_2(x_N, y_N)}{dx} & L & \frac{dV_J(x_N, y_N)}{dx} \\ \frac{dV_1(x_1, y_1)}{dy} & \frac{dV_2(x_1, y_1)}{dy} & L & \frac{dV_J(x_1, y_1)}{dy} \\ \frac{dV_1(x_2, y_2)}{dy} & \frac{dV_2(x_2, y_2)}{dy} & L & \frac{dV_J(x_2, y_2)}{dy} \\ \frac{dV_1(x_N, y_N)}{dy} & \frac{dV_2(x_N, y_N)}{dy} & L & \frac{dV_J(x_N, y_N)}{dy} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{dW(x_1, y_1)}{dx} \\ \frac{dW(x_2, y_2)}{dx} \\ \frac{dW(x_N, y_N)}{dx} \\ \frac{dW(x_1, y_1)}{dy} \\ \frac{dW(x_2, y_2)}{dy} \\ \frac{dW(x_N, y_N)}{dy} \end{bmatrix} \]

Again, conceptually easy to understand, although this can be relatively slow for high order fits.

\[ ZA = F \]
\[ Z^TZA = Z^TF \]
\[ A = (Z^TZ)^{-1}Z^TF \]
Discussion

- A priori knowledge should be used to determine the degree of fit needed for a given application.
- Orthogonality allows fast but noisy fits and can be used in cases where the gross features of the surface are to be analyzed.
- Least squares and Gram-Schmidt techniques provide more accurate fits, but are slower.