Ocular Wavefront Error Representation (ANSI Standard)

Jim Schwiegerling, PhD
Ophthalmology & Vision Sciences
Optical Sciences
University of Arizona
People

• Ray Applegate, David Atchison, Arthur Bradley, Charlie Campbell, Larry Horowitz, David Loshin, Scott MacRae, Susana Marcos, Dan Neal, Austin Roorda, Tom Salmon, Jim Schwiegerling, Larry Thibos, Rob Webb, David Williams and others...
Reasons for Standardization

- Zernike polynomials have traditionally been used to represent wavefronts in optical systems with circular pupils.
- First developed by Fritz Zernike in the 1930s for phase contrast microscopy. Multiple ways of scaling, orienting and ordering the basic polynomials have evolved, which can lead to confusion.
- The standard helps to remove ambiguity to facilitate clarity and exchange between all users.
- Lessons learned from Corneal Topography standards in the 1990s.
References


• ANSI Z80.28-2004 Methods for Reporting Optical Aberrations of Eyes.

• ISO/DIS 24157 (Draft currently being voted on 2007)
Wavefront Error Measurement
Coordinate System

Polar Coordinates:

\[ r = \sqrt{x^2 + y^2} \]

\[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \]

\( r \) ranges from \([0, \text{r}_{\text{max}}]\) \\
\( \theta \) ranges from \([0^\circ, 360^\circ]\)
Normalized Coordinate System

Normalized Polar Coordinates:

\[ \rho = \frac{r}{r_{\text{max}}} \]
\[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \]

\(\rho\) ranges from [0, 1]
\(\theta\) ranges from [0°, 360°]
Another Coordinate System

NON-STANDARD

Normalized Polar Coordinates:

\[ \rho = \frac{r}{r_{\text{max}}} \]

\[ \phi = \tan^{-1}\left(\frac{x}{y}\right) \]

\( \rho \) ranges from \([0, 1]\)

\( \theta \) ranges from \([-180^\circ, 180^\circ]\)
The Third Dimension

The z-axis is perpendicular to the x and y-axes and coincides with the Line of Sight.

The Line of Sight connects the point of fixation to the center of the entrance pupil, which in turn connects the center of the exit pupil to the fovea.
Zernike Polynomials

Radial Polynomial, ρ

Azimuthal Frequency, θ

$Z_0^0$, $Z_1^{-1}$, $Z_2^0$, $Z_2^2$, $Z_3^{-1}$, $Z_3^1$, $Z_4^{-2}$, $Z_4^2$, $Z_4^{-4}$, $Z_4^4$
In general, wavefronts can take on a complex shape and we need a way of breaking down this complexity into more simpler components.
Zernike Decomposition

\[ -0.003 \, \text{x} = + 0.002 \, \text{x} + 0.001 \, \text{x} \]

- Spherical Refractive Error
- Cylindrical Refractive Error
- Coma
Wavefront Error Representation

- The wavefront error can be represented by

\[ W(\rho, \theta) = \sum_{n,m} a_{n,m} Z_n^m (\rho, \theta) \]

\[ n = 0, 1, 2 \ldots ; m = -n, -n + 2 \ldots n-2, n \]
Generalized Zernike Polynomials

\[ Z_n^m (\rho, \theta) = \begin{cases} 
N_n^m R_n^{|m|} (\rho) \cos m\theta & ; \text{for } m \geq 0 \\
- N_n^m R_n^{|m|} (\rho) \sin m\theta & ; \text{for } m < 0 
\end{cases} \]

- Double Index
  - \( n \) is radial order
  - \( m \) is azimuthal frequency

- Normalization

- Radial Component

- Azimuthal Component
ANSI Standard Zernikes

constant that depends on \( n \) & \( m \)

Kronecker delta function

\[
\delta_{ab} = \begin{cases} 
1 & \text{for } a = b \\
0 & \text{for } a \neq b 
\end{cases}
\]

e.g.

\[
N_{3}^{-3} = \sqrt{\frac{2(3+1)}{1+0}} = \sqrt{8}
\]
ANSI Standard Zernikes

\[ R_n^{|m|}(\rho) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s! \left[ 0.5(n+|m|) - s \right] \left[ 0.5(n-|m|) - s \right]} \rho^{n-2s} \]

only depends on \(|m|\) (i.e. same for both sine & cosine terms)

Constant that depends on \(n\) and \(m\)

Powers of \(\rho\)

Factorial Function - \(a! = a(a-1)(a-2) \ldots (3)(2)(1)\)

\[ \text{e.g. } R_3^{-3}(\rho) = \sum_{s=0}^{(3-3)/2} \frac{(-1)^s (3-s)!}{s! [3-s]! [0-s]!} \rho^{3-2s} = \frac{3!}{3!} \rho^3 = \rho^3 \]
Example Zernike Polynomial

\[ Z^{-3}_3(\rho, \theta) = \begin{cases} 
N_n^m R_n^{|m|}(\rho) \cos m\theta & ; \text{for } m \geq 0 \\
-N_3^{-3} R_3^{-3}(\rho) \sin (-3\theta) & ; \text{for } m < 0
\end{cases} \]

\[ Z^{-3}_3(\rho, \theta) = \sqrt{8}\rho^3 \sin(3\theta) \]

Oblique Trefoil
Zernike Polynomials - Single Index

\[ Z_j \text{ where } j = \frac{n(n + 2) + m}{2} \]

ANSI STANDARD
Starts at 0
Left-to-Right
Top-to-Bottom

Radial Polynomial, \( \rho \)

Azimuthal Frequency, \( \theta \)
Other Single Index Schemes

NON-STANDARD

Starts at 1
cosines are even terms
sines are odd terms


Also Zemax “Standard Zernike Coefficients”
Other Single Index Schemes

NON-STANDARD

Starts at 1
increases along diagonal
cosine terms first
35 terms plus two extra
spherical aberration terms.
No Normalization!!!

Zemax “Zernike Fringe Coefficients”

Also, Air Force or University of Arizona
Other Single Index Schemes

- Born & Wolf
- Malacara
- Others???

Plus mixtures of non-normalized, coordinate systems.

Use two indices \( n, m \) to unambiguously define polynomials.
Use a single standard index if needed to avoid confusion.
**Examples**

**Example 1:**

0.25 D of myopia for a 4 mm pupil ($r_{max} = 2$ mm)

$$W = \frac{r^2}{8000} = \frac{(2\rho)^2}{8000} = \frac{\rho^2}{2000} = \frac{1}{4000} Z^0_0(\rho, \theta) + \frac{1}{4000\sqrt{3}} Z^0_2(\rho, \theta)$$
Example 2:

1.00 D of myopia for a 2 mm pupil ($r_{\text{max}} = 1 \text{ mm}$)

\[ W = \frac{r^2}{2000} = \frac{\rho^2}{2000} = \frac{1}{4000} Z_0^0(\rho, \theta) + \frac{1}{4000\sqrt{3}} Z_2^0(\rho, \theta) \]

Same Zernike Expansion as Example 1, but different $r_{\text{max}}$.

Always need to give pupil size with Zernike coefficients!!
Orders of Zernike Polynomials

Radial Polynomial, $\rho$

Low Order

Higher Order

Azimuthal Frequency, $\theta$

$Z_0^0$

$Z_1^{-1}$

$Z_1^1$

$Z_2^{-2}$

$Z_2^2$

$Z_3^{-3}$

$Z_3^3$

$Z_4^{-4}$

$Z_4^4$
Summary

• Use the two-index scheme $n, m$ to unambiguously define the Zernike polynomials whenever possible.
• If needed, use the standard single index top-to-bottom, left-to-right notation.
• Always include the size of the pupil with the Zernike coefficients.
• Be aware that other schemes exists in the literature and in other fields of optics.