Assessment of the accuracy of the crossed-cylinder aberroscope technique

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Simulations of the optics of the Howland crossed-cylinder aberroscope technique show that errors in alignment, data collection, and analysis can lead to unexpected asymmetries of the determined aberrations in a rotationally symmetric system. In particular, coma can be incorrectly indicated. The magnitude of the error in aberration measurement depends on the magnitude of the alignment, data collection, and alignment errors. These findings indicate that the tolerances for setting up the technique and data collection should be analyzed thoroughly before quantitative significance is given to the determined aberration coefficients. © 1998 Optical Society of America [S0740-3232(98)00709-1]


1. INTRODUCTION

The Howland crossed-cylinder aberroscope technique has been used several times to investigate the aberrations of the human eye.1–7 Results from most of these investigations show that the on-axis aberrations of the eye are far from being rotationally symmetrical. This is to be expected because the eye, being a biological system, is unlikely to be a rotationally symmetrical system and because the fovea is displaced from the optical axis. However, it is not certain how much of this asymmetry is due to errors in alignment of all the various components of the system. We became concerned after attempts to calibrate an aberroscope with a simple lens of known aberrations failed. We placed the lens accurately in the aberroscope and aligned it by ensuring that the point-spread function had no obvious asymmetries that could be due to coma or astigmatism. However, analysis of the aberroscope grid patterns showed asymmetrical spherical aberration and coma.

As a consequence, we became concerned that the construction of the aberroscope itself, its alignment to the eye, and errors in data collection and analysis may significantly influence the measured aberrations. We investigated these effects by computer modeling the optical properties of the crossed-cylinder aberroscope and eye and by systematically altering the model to simulate some potential sources of error, which are as follows:

Setting up the aberroscope:
Effect of the presence of the cross cylinder and its distance from the eye;
Whether the grid is undistorted or predistorted;
Errors in orientation of the crossed cylinder or grid, separately or in combination;
Centration of the crossed cylinder or grid, separately or in combination;
Grid spacing;
Number of sampling points.

Data collection and analysis:
Error in assumed grid spacing value,
Error in assumed vertex distance,
Choice of origin of measurement of coordinates,
Error in choice of grid center,
Effect of defocus of eye.

Choice of portion of grid and grid center,
Choice of grid retinal element size,
Errors in locating grid points,
Selection of only part of the retinal grid.

2. PRINCIPLE OF THE HOWLAND ABERROSCOPE

The detailed operation of the Howland aberroscope has been described in several publications.1–2,8 A description of some of the equations from these sources will serve as our starting point. A list of the important symbols used is given in Table 1.
Table 1. Important Symbols Used in Equations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>Distance from the cross cylinder to eye pupil (vertex distance +3.0 mm)</td>
</tr>
<tr>
<td>(K)</td>
<td>Defined by Eq. (9)</td>
</tr>
<tr>
<td>(\phi)</td>
<td>Angle of inclination of the positive cylinder axis to the X axis</td>
</tr>
<tr>
<td>(F_e)</td>
<td>Equivalent power of the eye (0.05994 mm(^{-1}) or 59.94 D(^{-1}) for the Le Grand eye used)</td>
</tr>
<tr>
<td>(F_c)</td>
<td>Power of the positive component of the crossed cylinder (0.005 mm(^{-1}) or 5 D used)</td>
</tr>
<tr>
<td>(W)</td>
<td>Wave aberration (in (\mu)m)</td>
</tr>
<tr>
<td>(\delta \xi', \delta \eta')</td>
<td>Transverse aberrations (in mm)</td>
</tr>
<tr>
<td>(W_1-W_{14})</td>
<td>Wave aberration coefficients (in (\mu)m/mm for (W_1) and (W_2, \mu)m/mm(^2) for (W_3), (W_4, \mu)m/mm(^3) for (W_5), and (\mu)m/mm(^4) for (W_{10}-W_{14}))</td>
</tr>
<tr>
<td>(X_g, Y_g)</td>
<td>Coordinates in the grid (in mm)</td>
</tr>
<tr>
<td>(X, Y)</td>
<td>Corresponding coordinates in the entrance pupil of the eye (in mm)</td>
</tr>
<tr>
<td>(X', Y')</td>
<td>Corresponding coordinates at the retina (in mm)</td>
</tr>
</tbody>
</table>

\(\text{*D, diopiter(s).}\)

The wave aberration \(W\) can be described by a polynomial in the intersection coordinates \(X\) and \(Y\) of a ray in the entrance pupil, of the form

\[
W(X, Y) = W_1 X + W_2 Y + W_3 X^2 + W_4 X Y + W_5 Y^2 + W_6 X^3 + W_7 X^2 Y + W_8 X Y^2 + W_9 Y^3 + W_{10} X^4 + W_{11} X^3 Y + W_{12} X^2 Y^2 + W_{13} X Y^3 + W_{14} Y^4 + \text{higher-order terms},
\]

(1)

which was the form suggested by Howland and Howland,\(^1,2\) but with symbols \(B, C, \ldots, O\) instead of \(W_i\) to \(W_{14}\). For any optical system a ray through the point \((X, Y)\) has the transverse aberrations \((\delta \xi', \delta \eta')\), in the image plane, given by the relations

\[
\delta \xi' = \frac{1}{F_e} \frac{\partial W(X, Y)}{\partial X}, \quad \delta \eta' = \frac{1}{F_e} \frac{\partial W(X, Y)}{\partial Y},
\]

(2)

where \(F_e\) is the equivalent power of the optical system (in our case, the eye).

Substituting for \(W(X, Y)\) from Eq. (1) into relations (2), but neglecting the higher-order terms, gives

\[
-\delta \xi' F_e = W_1 + 2 W_3 X + W_4 Y + 3 W_6 X^2 + 2 W_7 X Y + W_8 Y^2 + 4 W_{10} X^3 + 3 W_{11} X^2 Y + 2 W_{12} X Y^2 + W_{13} Y^3,
\]

(3a)

\[
-\delta \eta' F_e = W_2 + W_4 X + 2 W_5 Y + W_7 X^2 + 2 W_8 X Y + 3 W_9 Y^2 + W_{11} X^3 + 2 W_{12} X^2 Y + 3 W_{13} X Y^2 + 4 W_{14} Y^3.
\]

(3b)

Thus, if we can measure the transverse aberrations for a number of rays, Eqs. (3a) and (3b) give us a set of simultaneous equations in which we have to solve for the \(W_i\) to \(W_{14}\) coefficients. In the aberroscope technique there is not an equal number of unknowns to equations, so the usual routines used to solve a set of simultaneous equations with an equal number of unknowns cannot be used. Two alternative methods are the original orthogonal method of Howland and Howland\(^1,2\) and the least-squares method of Smith \(et\al.\)^8

In the case of a rotationally symmetrical system, the on-axis monochromatic aberration is purely spherical aberration, and we can express the corresponding wave aberration as a function of only one variable, the ray height in the pupil, which we will arbitrarily choose as \(Y\). Thus for a distance \(Y\) from the optical axis, \(W(Y)\) can be fitted by a polynomial of the form

\[
W(Y) = W_{4,0} Y^4 + \text{higher-order terms}
\]

(4)

In this case the equation corresponding to Eqs. (3a) and (3b) is

\[
-\delta \eta' F_e = dW(Y)/dY = 4 W_{4,0} Y^3 + \text{higher-order terms}
\]

(4a)

The primary spherical aberration coefficient \(W_{4,0}\) is related to the terms \(W_{10}, W_{12},\) and \(W_{14}\) in Eq. (1) by

\[
W_{10} = W_{14} = 0.5 W_{12} = W_{4,0}.
\]

(5a)

Also,

\[
W_{11} = W_{13} = 0.
\]

(5b)

The theory described above applies to any optical system. However, in practice there are difficulties in implementing the method because the transverse aberrations are small and it is difficult to match any particular ray in the image plane with the corresponding ray in the entrance pupil. In the aberroscope technique, placing a crossed cylinder in front of the system spreads out the ray pattern in the image plane in a controlled manner, allowing both ease of measurement and ray identification. It has been assumed that the presence of the crossed cylinder has no effect on the aberrations of the main system, but we will show that the crossed cylinder does have a small effect on these aberrations even when the system is perfectly aligned and can have large effects if the system is misaligned.

In the treatment given below, wave aberrations are measured in micrometers because they are small, and pupil coordinates are given in millimeters because this is convenient.

3. METHOD

A computer program (TRACE.PAS)\(^9\) was written in Pascal by G. Smith to trace rays through a schematic crossed-cylinder–eye system. The chosen eye was the Le Grand schematic eye,\(^10\) but with its surfaces aspherized to give a primary spherical aberration similar to that in real eyes. Since this model is rotationally symmetrical about the optical axis, the wave aberration can be described by Eqs. (4), but only the primary coefficient \(W_{4,0}\) was considered.
There is no definitive value of $W_{4,0}$ for real eyes because it varies greatly among individuals and possibly with the method of measurement. Using older data in the literature, Smith and Atchison estimated the population mean to be $0.025 \text{ mm/mm}^4$, and, from more recent aberroscope results, they estimated the population mean to be much lower, at approximately $0.009 \text{ mm/mm}^4$. The larger value was taken here because the Smith–Atchison survey of published values shows that there is a large variation in spherical aberration among the population and that the larger values will show error effects in the aberroscope technique more readily than will the lower values, and unnatural aberrations being induced by refractive surgery into the corneas of eyes that we are currently measuring are more than 30 times larger than are naturally occurring aberrations.

The data for this crossed-cylinder–eye system are shown in Table 2. Various parameters can be changed to simulate the potential sources of error listed in Section 1. For each error condition a grid of rays was traced through the system, and the intersection coordinates ($d_x$, $d_y$) of each ray with the image plane (retina) were calculated. Figure 1 shows a typical result. These coordinates are affected by (i) conventional aberrations (which we seek to measure), (ii) the choice of origin, and (iii) the presence of the crossed cylinder. Fortunately, as we show below, we can separate these different contributions. The choice of coordinate system origin affects only the values of $W_1$ and $W_2$, and the presence of the crossed cylinder has predictable effects on the values of $W_3$ to $W_5$.

The transverse aberrations results were then fed into a program called WAB.PAS, written in Pascal by G. Smith, which calculated the wave aberration coefficients ($W_1$ to $W_{14}$) by the least-squares method described by Smith et al. The values of the aberration coefficients could then be compared with the expected values.

There are two types of expected aberration values: those that are due to the eye alone, and those that will be added because of the presence of the crossed cylinder. In the following discussion we distinguish between these two types.

### A. Expected Aberration Values of the Eye Alone

These are the aberrations of the model eye in the absence of the crossed cylinder. Since the system is rotationally symmetrical, the only aberration will be spherical aberration, which can be described by Eqs. (4). A third program, FAP.PAS, written by G. Smith, was used to calculate the wave aberration for a number of rays, and the resulting aberrations are given in Table 3. For each ray, the ray height $Y$ and the corresponding wave aberration were fitted to Eqs. (4), and a least-squares method was used to solve for the aberration coefficients. The results are given in Table 4. To check the accuracy of relations (2), which we note are approximate, the transverse aberration data were fitted to Eq. (4a), and the corresponding coefficients are given in Table 4. Note that the two sets of values are very similar, showing that, at least for this model eye, relations (2) are sufficiently accurate.

Substituting the wave-aberration-derived value of the coefficient $W_{4,0}$ given in Table 4 into Eq. (5a) gives the corresponding coefficients of the Howland polynomial as

$$W_{10} = W_{14} = 0.024701 \text{ mm/mm}^4,$$  \hspace{1cm} (6a)
\[ W_{12} = 0.048142 \mu m/mm^2. \]  

We could have taken the value of 0.024104 derived from the transverse aberrations rather than 0.024071, but, as there is so little difference, the choice is not important.

For a rotationally symmetrical system on axis there should be no coma or astigmatism, and therefore all the coefficients from \( W_3 \) to \( W_5 \) should be zero. That is,

\[ W_3 = W_4 = W_5 = W_6 = W_7 = W_8 = W_9 = 0. \]  

(7)

B. Expected Aberration Values Due to the Presence of the Crossed Cylinder

We now show that, for the schematic eye, the presence of the crossed cylinder affects the coefficients \( W_3 \), \( W_4 \), and \( W_5 \). Their values will depend on any defocus, the power and orientation of the crossed-cylinder lenses, and the distance between the crossed cylinder and the eye. The values of these coefficients due to the presence of the crossed cylinder can be predicted from theory. We can do this by tracing suitable paraxial rays through the system and finding the crossing points on the retina. Smith et al.\(^8\) showed (from their Eq. 29) that a paraxial ray passing through a point \((X_g, Y_g)\) at the crossed cylinder meets the retina at the points \((X', Y')\) given by the equations

\[ X' = K(F_c/F_e)Y_g, \]  

(8)

\[ Y' = K(F_c/F_e)X_g, \]  

(9)

where \( K = 2 \sin(\phi) \cos(\phi) \), \( \phi \) is the angle of inclination of the positive cylinder axis to the \( X \) axis, and \( F_e \) is the power of the positive cylindrical lens. However, the coordinates \((X, Y)\) in Eq. (1) are the corresponding coordinates in the eye pupil, which do not have the same values as the coordinates \((X_e, Y_e)\) at the grid. Smith et al.\(^8\) showed (from their Eq. 23) that these are related by the equations

\[ X = X_g + dKF_cY_g, \]  

\[ Y = Y_g + dKF_cX_g, \]  

(10)

where \( d \) is the distance from the crossed cylinder to the front principal plane of the eye. We can transform these equations to obtain

\[ X_g = \frac{(dKF_cY - X)}{[(dKF_c)^2 - 1]}, \]  

(10a)

\[ Y_g = \frac{(dKF_cX - Y)}{[(dKF_c)^2 - 1]}. \]  

(10b)

From Eqs. (8) and (10) we have

\[ X' = \frac{KF_c(dKF_cX - Y)}{F_e [(dKF_c)^2 - 1]}, \]  

(11a)

\[ Y' = \frac{KF_c(dKF_cY - X)}{F_e [(dKF_c)^2 - 1]}. \]  

(11b)

These quantities are the “transverse aberrations” induced by the presence of the crossed cylinder and so are equivalent to the quantities \((\delta \xi', \delta \eta')\) in relations (2). Thus from relations (2) we have

\[ KF_c \frac{(dKF_cX - Y)}{F_e [(dKF_c)^2 - 1]} = -\frac{1}{F_e} \frac{\partial}{\partial X} [W_3X^2 + W_4XY + W_5Y^2], \]  

(12a)

\[ KF_c \frac{(dKF_cY - X)}{F_e [(dKF_c)^2 - 1]} = -\frac{1}{F_e} \frac{\partial}{\partial Y} [W_3X^2 + W_4XY + W_5Y^2]. \]  

(12b)

Solving these equations for the coefficients \( W_3 \), \( W_4 \), and \( W_5 \) leads to the solutions

\[ W_3 = \frac{d(KF_c)^2}{2[1 - (dKF_c)^2 - 1]}, \]  

(13a)

\[ W_4 = \frac{KF_c}{1 - (dKF_c)^2}, \]  

(13b)

\[ W_5 = \frac{W_3}{2[1 - (dKF_c)^2]}. \]  

(13c)

We give some numerical values for these quantities further below.

Finally, the values of the coefficients \( W_1 \) and \( W_2 \) depend only on the coordinate values of the chosen grid center and should be zero and independent of the aberrations of the system under investigation. Any nonzero values imply that the grid center is not at \((0, 0)\) in the coordinate system used.

4. RESULTS

The aberration coefficients of Eq. (1) were determined by use of a \(5 \times 5\) grid with a grid spacing of 1 mm and with

<table>
<thead>
<tr>
<th>Ray Height (mm)</th>
<th>Wave Aberration (µm)</th>
<th>Transverse Aberration (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y )</td>
<td>( W(Y) )</td>
<td>( \delta \eta(Y') )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.00151</td>
<td>-0.000200</td>
</tr>
<tr>
<td>1.0</td>
<td>0.02417</td>
<td>-0.001614</td>
</tr>
<tr>
<td>1.5</td>
<td>0.12311</td>
<td>-0.005486</td>
</tr>
<tr>
<td>2.0</td>
<td>0.39225</td>
<td>-0.013125</td>
</tr>
<tr>
<td>2.5</td>
<td>0.96739</td>
<td>-0.025920</td>
</tr>
<tr>
<td>3.0</td>
<td>2.027975</td>
<td>-0.045329</td>
</tr>
<tr>
<td>3.5</td>
<td>3.08076</td>
<td>-0.072794</td>
</tr>
<tr>
<td>4.0</td>
<td>6.57989</td>
<td>-0.109444</td>
</tr>
<tr>
<td>4.5</td>
<td>10.64560</td>
<td>-0.155224</td>
</tr>
<tr>
<td>5.0</td>
<td>16.26265</td>
<td>-0.206139</td>
</tr>
</tbody>
</table>

Table 3. Wave and Transverse Aberrations at Different Ray Heights from FAP.PAS, for the Eye Data Given in Table 2

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>From ( W(Y) )</th>
<th>From ( \delta \eta(Y') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{12} ) (µm/mm²)</td>
<td>0.024071</td>
<td>0.024104</td>
</tr>
<tr>
<td>( W_{23} ) (µm/mm²)</td>
<td>0.000111</td>
<td>0.000079</td>
</tr>
<tr>
<td>( W_{30} ) (µm/mm²)</td>
<td>0.000001</td>
<td>0.000001</td>
</tr>
</tbody>
</table>
the grid placed at various vertex distances. In most conditions a vertex distance of 30 mm was used.

In addition to these conditions, the grid was used in both undistorted (i.e., rectangular) and predistorted shapes. The original programming assumed that the grid was projected square onto the pupil, necessitating use of a grid that is distorted in shape according to the vertex distance of the grid. By contrast, the least-squares method presented and described by Smith et al. does not require a predistorted grid. All that is required is a knowledge of the effective grid points in the pupil, and these can be found from a knowledge of the actual grid shape and vertex distance, with Eqs. (10) and (10a). As a result, the program TRACE.PAS can use either an undistorted grid or a predistorted grid, and if the grid is predistorted it determines the actual true grid points from the vertex distance. If the predistorted grid is chosen, the 1-mm spacing applies at the entrance pupil of the eye. That is, the grid projected onto the pupil of the eye is square, with a side spacing of 1 mm.

In the analysis we have divided the errors into (a) those that are due to the aberroscope setup, and (b) those that occur at data collection.

A. Errors Due to the Aberroscope Setup

1. Effect of the Presence of the Crossed Cylinder and Its Distance from the Eye

Equations (13) show that the presence of the crossed cylinder and its vertex distance from the eye affect the coefficients $W_3$, $W_4$, and $W_5$. Because the path of rays through the eye is affected by the crossed cylinder, the existing aberrations $W_{10}$ to $W_{14}$ are also affected. However, the coma coefficients are zero, as expected.

Figure 2 shows the effects of vertex distance on the aberration coefficients of the crossed-cylinder–eye system for an undistorted grid, for a predistorted grid, and for the model eye alone. Figures 2(a) and 2(b) show the values of $W_3$, $W_4$, and $W_5$, including expected values for the crossed-cylinder–eye system. The expected values of the coefficients are very close to the measured values. Figures 2(c)–2(e) show the values of the spherical aberration terms $W_{10}$, $W_{11}$, $W_{12}$, $W_{13}$, and $W_{14}$, with values due to the model eye alone being given by Eqs. (5). Note that the scales of the figures are expanded to illustrate the errors that are induced.

In general, the aberration coefficients are not correct for any vertex distance, with the error depending on the aberration term, the vertex distance, and whether the grid has been predistorted. The coefficients $W_{10}$, $W_{12}$, and $W_{14}$ for the predistorted grid appear to be more sensitive to vertex distance than those for the undistorted grid, and this result is discussed below, so we restrict the immediate discussion to those of the undistorted grid. The values of $W_2$, $W_4$, and $W_5$ are close to the values expected to be induced by the crossed cylinders, with the values of $W_3$ and $W_4$ being less in error than $W_3$. The values of $W_{10}$, $W_{12}$, and $W_{14}$ are greater than expected by approximately 5% and show similar and slight dependencies on vertex distance. The coefficients $W_{11}$ and $W_{13}$, while small, should be zero.

2. Effect of Predistortion of the Grid

Differences are expected between the results for undistorted and those for predistorted grids because the two situations lead to different ray paths through the crossed-cylinder–eye system, and hence the rays will pick up different amounts of aberration. Figure 2 shows these differences. In particular, the results seem to indicate that a predistorted grid leads generally to greater changes with vertex distance. The reasons for this are subtle; the most important seems to be the choice of the unaberrated grid size, which we discuss in Subsection 4.B.

3. Errors in the Orientation of the Crossed-Cylinder–Grid Axes Combination

The procedure for calculating the wave aberration coefficients requires a knowledge of the orientation of the cross-cylinder lenses. In this study we have assumed that the angles of orientation are those given in Table 2. Once assembled as a single unit, they will all have the same rotation error if they are incorrectly placed in the aberroscope mounting. However, when the crossed-cylinder–grid system is being assembled, independent rotation and orientation errors may occur in each component.

Table 5 shows the aberration values for the crossed-cylinder–grid system rotated through $+5^\circ$; that is, the positive cylinder axis is $50^\circ$ to the X axis, the negative cylinder is $-40^\circ$ to that axis, and the grid is also rotated through $5^\circ$. We can see that this orientation error affects the nonzero aberrations. There are negligible changes in $W_{10}$, $W_{12}$, and $W_{14}$ but greater changes in $W_3$ and $W_5$ and in $W_{11}$ and $W_{13}$.

Perhaps the most interesting result is that an error in orientation of the crossed cylinder leads to asymmetric aberrations, in both astigmatism and spherical aberration. However, this type of error does not induce any coma.

Table 5 also shows the effect of a $5^\circ$ rotational error in the crossed cylinder and the grid separately. In this case the effects are much larger, and now the changes in $W_{10}$ and $W_{14}$ are in the region of 7%. Once again, these rotation errors lead to asymmetry in the aberrations.

4. Centration Error of the Crossed-Cylinder–Grid Unit

Let us assume now that the axis of the crossed-cylinder–grid is not aligned perfectly with the optical–visual axis of the eye; that is, the two axes are displaced laterally by a certain distance. Table 5 shows the effect of decentering the crossed-cylinder–grid unit upward by $+1$ mm under three conditions: the crossed cylinder alone, the grid alone, and both together. The reference for these results is the retinal intersection of the ray passing through the grid center. The results in this table show small changes in spherical aberration and a small asymmetry. A considerable amount of coma is induced when the crossed-cylinder–grid unit or the grid alone is decentered. The magnitudes of the effects depend on the level of the spherical aberration in the eye, and, if there were no spherical aberration, no coma would be induced.

Most of the induced coma can be attributed to a shift in the choice of effective pupil center rather than to a shift in the choice of position of the grid. We can demonstrate
this as follows. In Eq. (1) we change the pupil center to \( Y = Y_0 \), so that the spherical aberration terms become

\[
W_{10}X^4 + W_{11}X^3(Y - Y_0) + W_{12}X^2(Y - Y_0)^2 \\
+ W_{13}X(Y - Y_0)^3 + W_{14}(Y - Y_0)^4.
\]

(14)

Expanding the right-hand side of this expression, we have coma terms as follows:

\[
-W_{11}Y_0X^3, \quad -W_{12}2Y_0X^2Y, \\
-W_{13}3Y_0XY^2, \quad -W_{14}4Y_0Y^3.
\]

(15)

which can be written as

\[
W_6X^3, \quad W_7X^2Y, \quad W_8XY^2, \quad W_9Y^3.
\]

(16)

The results shown in Table 5 for the shift of the crossed-cylinder–grid system are close to those predicted by these equations, which shows that this type of error is equivalent to a pupil decentration with a small residual asymmetric effect that is probably due to the effect of the crossed cylinder. Note that, while the numerical values for induced coma shown in Table 5 are close to those predicted by the above equations, the sign is opposite.

We have noted that the subjective appearance of aberroscope grid patterns changes very little with crossed-

Fig. 2. Effect of vertex distance on the expected values (crossed-cylinder–eye system) of aberration coefficients for undistorted and predistorted grids. The vertex distance is measured to the front vertex of the eye and not to the entrance pupil, which is 3.0 mm farther inside the eye. (a) \( W_3 \) and \( W_5 \), (b) \( W_4 \), (c) \( W_{10} \) and \( W_{14} \), (d) \( W_{11} \) and \( W_{13} \), (e) \( W_{12} \).
cylinder–grid unit decentration, a result that seems to be at odds with the change in coma found in these simulations. This can be explained by the eye’s changing reference from the ray passing through the center of the (decentered) grid to that passing through the center of the pupil. When the reference ray is changed from the grid axis to the eye axis, most of the changes in aberrations due to centration errors of the crossed-cylinder–grid combination will disappear.

### B. Errors Occurring at Data Collection

#### 1. Error in Assumed Grid Spacing Value

The grid spacing is required for calculation of the aberration coefficients, and, if the value used is in error, it will lead to errors in the calculated values of the coefficients. The aberration coefficients resulting from assuming a grid spacing of 1.1 mm rather than the actual spacing of 1 mm are shown in Table 6. Comparing the results with the centered values shows that the +10% error in assumed grid spacing leads to a reduction of approximately 17% in the spherical aberration terms.

#### 2. Error in Assumed Vertex Distance

The grid vertex distance is also required for the calculations. The aberration coefficients that result from assuming a vertex distance of 25 mm rather than from the actual vertex distance of 30 mm are shown in Table 6. Comparing the results with the centered values shows that the 5-mm underestimate of vertex distance changes the values of \( W_{10} \), \( W_{12} \), and \( W_{14} \) by less than 0.5%. The fractional changes in \( W_{11} \) and \( W_{13} \) are much larger at 57%.

#### 3. Choice of Origin for Measurement of Coordinates

The program TRACE.PAS, which calculates the transverse aberrations, calculates all the transverse aberrations relative to the origin \((0, 0)\), which is assumed to be on the optical axis. The program WAB.PAS uses these same values. If instead we take values from a real image, say, from a video screen, the origin for all the transverse aberrations will usually be one corner of the screen, although the software can shift this to some other point. The choice of this center affects only the values of \( W_1 \) and \( W_2 \).

#### 4. Error in Choice of Grid Center

In the calculation of the wave aberration coefficients from the transverse aberrations it is necessary to correctly associate each transverse aberration with the corresponding ray in the pupil. If the complete grid is visible this association is not difficult and is achieved simply by location of the center of the grid. However, if an incorrect choices made, there will be resulting errors in the wave aberration coefficient values. To show this effect, we misplaced the grid center by one grid element upward. The results are shown in Table 6. It is clear that such an error induces considerable coma but has no effect on the
5. Effect of Defocus

Errors in defocus (refractive error) should cause changes only in the wave aberration coefficients $W_3$ and $W_5$, and these changes should be equal and related to the defocus by the equation

$$\Delta W_3 = \Delta W_5 = \frac{\delta F}{2} = \frac{F^2 \delta l'}{2n'}.$$ 

With $F = 59.940$ D = 0.059940 mm$^{-1}$, and $n' = 1.336$, we have

$$\Delta W_3 = \Delta W_5 = 0.0013446 \delta l',$$  \hfill (18)

which are added to the values of $W_3$ and $W_5$ from other sources. Table 6 contains the wave aberration data for the schematic eye given an axial length 0.371854 mm too long, simulating approximately $-1$ D of myopia. As expected, defocus has substantial effects on only $W_3$ and $W_5$. The increases are within 2% of those predicted by relation (18).

Small changes have occurred in the other values because of the effect of the defocus on the estimated size of the unaberrated grid square. We can see from Table 6 that the best estimate of unaberrated retinal grid size is a little less than the correct value, which in turn will lead to the prediction of slightly larger aberrations.

### Table 6. Comparison of the Expected and Centered Crossed-Cylinder Aberration Coefficient Values and Actual Values for Various Sources of Error in the Data Collection and Analysis

<table>
<thead>
<tr>
<th>Error Source</th>
<th>Expected</th>
<th>Centered Crossed Cylinder</th>
<th>Assumed Grid Size, 1.1 mm</th>
<th>Assumed Vertex Distance, 25 mm</th>
<th>Error in Choice of Grid Center, 1 grid vertical$^b$</th>
<th>Defocus, $\sim 1$ D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-4.995577</td>
<td>0</td>
</tr>
<tr>
<td>$W_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.093102</td>
<td>0</td>
</tr>
<tr>
<td>Astigmatism</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_3$</td>
<td>0.424023</td>
<td>0.418905</td>
<td>0.418905</td>
<td>0.351709</td>
<td>0.476019</td>
<td>0.929524</td>
</tr>
<tr>
<td>$W_4$</td>
<td>-5.139676</td>
<td>-5.156068</td>
<td>-5.156068</td>
<td>-5.120951</td>
<td>-5.108157</td>
<td>-5.262921</td>
</tr>
<tr>
<td>$W_5$</td>
<td>= $W_3$</td>
<td>= $W_3$</td>
<td>= $W_3$</td>
<td>= $W_3$</td>
<td>0.574018</td>
<td>$W_3$</td>
</tr>
<tr>
<td>Comalike terms</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.021351</td>
<td>0</td>
</tr>
<tr>
<td>$W_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.103661</td>
<td>0</td>
</tr>
<tr>
<td>$W_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.030808</td>
<td>0</td>
</tr>
<tr>
<td>$W_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.101714</td>
<td>0</td>
</tr>
<tr>
<td>Spherical-aberration-like terms</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_{10}$</td>
<td>0.024071</td>
<td>0.025235</td>
<td>0.020855</td>
<td>0.025129</td>
<td>0.025235</td>
<td>0.026164</td>
</tr>
<tr>
<td>$W_{11}$</td>
<td>0</td>
<td>0.004896</td>
<td>0.003881</td>
<td>0.008833</td>
<td>0.004896</td>
<td>0.004719</td>
</tr>
<tr>
<td>$W_{12}$</td>
<td>0.048142</td>
<td>0.050668</td>
<td>0.041875</td>
<td>0.050515</td>
<td>0.050668</td>
<td>0.052577</td>
</tr>
<tr>
<td>$W_{13}$</td>
<td>0</td>
<td>= $W_{11}$</td>
<td>= $W_{11}$</td>
<td>= $W_{11}$</td>
<td>= $W_{11}$</td>
<td>$W_{11}$</td>
</tr>
<tr>
<td>$W_{14}$</td>
<td>0.024071</td>
<td>= $W_{10}$</td>
<td>= $W_{10}$</td>
<td>= $W_{10}$</td>
<td>= $W_{10}$</td>
<td>$W_{10}$</td>
</tr>
<tr>
<td>Grid size (on retina)</td>
<td>0.083416</td>
<td>0.083296</td>
<td>0.083296</td>
<td>0.083296</td>
<td>0.083296</td>
<td>0.083078</td>
</tr>
</tbody>
</table>

$^a$ All results are for a $5 \times 5$ undistorted grid, with a 1-mm grid spacing, placed 30 mm from the anterior corneal surface of the eye.

$^b$ By one grid element upward.
tances and for both the undistorted and the predistorted grids. The results are plotted in Fig. 3. The undistorted grid gives more reliable estimates than the predistorted grid. In this example the predistorted grid gives larger values of the unaberrated grid size. This in turn leads to the lower aberrations shown in Fig. 2.

7. Errors in Locating Grid Points

Errors inherent in choosing the grid points will lead to an uncertainty in the final wave aberration coefficients. To examine these types of errors, WAB.PAS was programmed to introduce uncertainties in the positions of grid points by means of uniform probability functions in the horizontal and vertical transverse aberrations. Table 7 shows the results for 50 simulations, with the uncertainty being taken as 1/10 of a grid element width for a 1° off-axis angle to show the effect on coma. The uncertainty can induce an apparent level of asymmetric coma. While the mean values for 50 simulations are close to the expected values, the standard deviations are high and vary from 17% to 315% of the mean values.

8. Selection of Only Part of the Grid and the Choice of Center of This Subgrid

Frequently the complete grid is not visible, because of vignetting by the iris, or is not usable because of gross distortions or poor image quality. In these cases we use only a portion of the grid. To investigate the effect of using only a part of the whole grid, we used a 7 × 7 grid with 1-mm grid spacing instead of the 5 × 5 grid used so far. The results of choosing only part of the 7 × 7 grid are shown in Table 8. We used two subgrids, a 5 × 5 grid and a 4 × 4 grid, both taken from the bottom right-hand corner of the full grid. For the larger of these two subgrids there is a decrease in the spherical terms, as high as 4% in the value of \( W_{12} \). Surprisingly, a small amount of coma appears, but this is much smaller than the spherical aberration. If we look at the values for the smaller of the two subgrids, the effects are much more pronounced. The coma coefficients are now comparable with the spherical aberration coefficients. The changes in spherical aberration and appearance of coma can be explained as follows.

(1) There is an effect due to the change in estimate of the unaberrated retinal grid size. There is a clear difference in this value between the two subgrids.

(2) If we take only one half of a symmetrical function that is not well fitted by the spherical-aberration-

Table 7. Effect of Uncertainty on Reading the Grid Points, Assuming That the Uncertainty on Each Grid Point Is Defined by a Uniform Probability Distribution with Extreme Values Being 1/10 the Width of a Grid Element

<table>
<thead>
<tr>
<th>Error Source</th>
<th>Correct</th>
<th>Mean</th>
<th>SD(^a)</th>
<th>SD/Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Psi_6 )</td>
<td>0.002902</td>
<td>0.003838</td>
<td>0.012083</td>
<td>3.149</td>
</tr>
<tr>
<td>( \Psi_7 )</td>
<td>0.068138</td>
<td>0.068960</td>
<td>0.016681</td>
<td>0.243</td>
</tr>
<tr>
<td>( \Psi_8 )</td>
<td>0.010036</td>
<td>0.014507</td>
<td>0.014503</td>
<td>1.000</td>
</tr>
<tr>
<td>( \Psi_9 )</td>
<td>0.069022</td>
<td>0.068387</td>
<td>0.011445</td>
<td>0.167</td>
</tr>
<tr>
<td>( \Psi_{10} )</td>
<td>0.025568</td>
<td>0.026720</td>
<td>0.009772</td>
<td>0.366</td>
</tr>
<tr>
<td>( \Psi_{11} )</td>
<td>0.004617</td>
<td>0.003933</td>
<td>0.010625</td>
<td>2.702</td>
</tr>
<tr>
<td>( \Psi_{12} )</td>
<td>0.051649</td>
<td>0.054091</td>
<td>0.009390</td>
<td>0.174</td>
</tr>
<tr>
<td>( \Psi_{13} )</td>
<td>0.004965</td>
<td>0.004020</td>
<td>0.010571</td>
<td>2.629</td>
</tr>
<tr>
<td>( \Psi_{14} )</td>
<td>0.025936</td>
<td>0.026822</td>
<td>0.008940</td>
<td>0.333</td>
</tr>
</tbody>
</table>

\(^a\)The means and standard deviations were calculated from 50 repeated simulations. The aberrations were calculated for a point 1° off axis.

Standard deviation.

Table 8. Comparison of the Expected and Centered Crossed-Cylinder Aberration Coefficient Values and Actual Values for Various Sources of Error in the Data Collection and Analysis

<table>
<thead>
<tr>
<th>Error Source</th>
<th>Expected Centered Crossed Cylinder Subgrid (3, 3) to (7, 7) (4, 4) to (7, 7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>( \Psi_1 ) 0 0 0.018897 (-0.033052)</td>
</tr>
<tr>
<td>( \Psi_2 ) 0 0 (-0.002164) (-0.012411)</td>
<td></td>
</tr>
<tr>
<td>Astigmatism</td>
<td>( \Psi_3 ) 0.424023 0.421954 0.421961 0.389221</td>
</tr>
<tr>
<td>( \Psi_4 ) (-5.138676) (-5.147680) (-5.155507) (-5.366720)</td>
<td></td>
</tr>
<tr>
<td>( \Psi_5 ) (-\Psi_3) (-\Psi_3) (-\Psi_3) (-\Psi_3)</td>
<td></td>
</tr>
<tr>
<td>Comalike terms</td>
<td>( \Psi_6 ) 0 0 (-0.002164) (-0.012411)</td>
</tr>
<tr>
<td>( \Psi_7 ) 0 0 (-0.005883) (-0.036358)</td>
<td></td>
</tr>
<tr>
<td>( \Psi_8 ) 0 0 (-\Psi_7) (-\Psi_7)</td>
<td></td>
</tr>
<tr>
<td>( \Psi_9 ) 0 0 (-\Psi_8) (-\Psi_8)</td>
<td></td>
</tr>
<tr>
<td>Spherical-aberration-like terms</td>
<td>( \Psi_{10} ) 0.024071 0.025429 0.025068 0.024914</td>
</tr>
<tr>
<td>( \Psi_{11} ) (-0.003675) (-0.003480) (-0.001006)</td>
<td></td>
</tr>
<tr>
<td>( \Psi_{12} ) 0.048142 0.050568 0.048516 0.045002</td>
<td></td>
</tr>
<tr>
<td>( \Psi_{13} ) 0 (-\Psi_{11}) (-\Psi_{11}) (-\Psi_{11})</td>
<td></td>
</tr>
<tr>
<td>( \Psi_{14} ) 0.024071 (-\Psi_{10}) (-\Psi_{10}) (-\Psi_{10})</td>
<td></td>
</tr>
<tr>
<td>Grid size 0.083416 0.083296 0.083296 0.081375</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\)All results are for a 7 × 7 undistorted grid, with a 1-mm grid spacing, placed 30 mm from the anterior corneal surface of the eye. The subgrid values were the same as those for a whole grid with the corresponding weights set to zero.
like terms because of higher-order terms, a coma term will appear that would not be present if the full symmetrical function were analyzed. This hypothesis was tested by curve fitting to the wave aberration values given in Table 3. A polynomial containing a \( W_{37} r^3 \) term and a \( W_{40} r^4 \) term fitted to the wave aberration data for \( r = -2.5 \) to +2.5 mm gave \( W_{37} = 0 \) and \( W_{40} = 0.024723 \mu \text{m/mm}^3 \). The same function fitted to only one half the data gave \( W_{37} = -0.002033 \mu \text{m/mm}^3 \) and \( W_{40} = 0.025574 \mu \text{m/mm}^3 \). The presence of the coma term provides a better fit than the spherical aberration term alone, because of the presence of the higher-order spherical aberration.

(3) When the full grid is not visible and clear there will be some uncertainty in deciding which grid point corresponds to the center of the pupil. In the above analysis we knew which was the pupil center and used that value. In a real situation it may not be obvious. For a healthy eye with typical levels of aberration and no obvious optical pathology (e.g., keratoconus, nuclear cataract) we would expect the least distorted grid to be the central grid. However, there will be some uncertainty, especially if the eye has such pathology, which may lead to an incorrect choice, which in turn will lead to errors in the calculated aberrations. The effect of choosing the wrong grid center (one grid point in a vertical direction) was discussed above.

5. DISCUSSION AND CONCLUSIONS

When properly aligned, the Howland 5-D cross-cylinder aberroscope introduces small and probably unimportant errors in measured aberrations. In contrast, larger errors in measured aberrations will occur from the incorrect choice of grid center and from errors in setting up the aberroscope. As an example, a 1-mm error in choice of grid center produces a coma error with a larger coefficient than that of the existing spherical aberration.\(^{13}\)

Failing to properly identify the true center of the pupil affects all aberration measurement schemes. Howland and Howland\(^2\) were aware of this potential problem and incorporated in their early programming the selection of the pupil center by the observer, since the pupil center did not need to be at the intersection of any grid lines or in the center of any grid unit.

As a rule, alignment errors and data collection–analysis errors tend to increase the measured aberrations and tend to affect the asymmetrical aberrations more than the symmetrical aberrations. In the case of highly aberrated eyes, such as in keratoconus or after refractive surgery, it is probably wise to place more emphasis on the qualitative result and less on the quantitative result.

It is not possible to predict the expected errors in any particular setup a priori because the level of the resulting errors in the measured aberrations depends on the magnitude and the source of the alignment error and on the magnitude of the actual aberration of the eye. Therefore we recommend that users of the aberroscope, and of any similar system, carefully analyze their setup by a similar optical simulation before quantitative significance is given to any of the measured aberration values. Furthermore, we recommend that users of any aberration measurement system set up a routine calibration protocol using schematic eyes with known aberrations. If eyes with large aberrations are to be measured, it would probably be wise to calibrate with schematic eyes having similarly large aberrations. To reduce the problems of induced coma, caused by incorrect choice of pupil center, we recommend that a reference marker that can easily be identified in the retinal image of the grid be placed at the center of the actual grid and that a pupil monitoring system be installed to ensure that the central ray of the reference marker actually passes through the center of the eye’s pupil.

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REFERENCES AND NOTES

9. Please contact George Smith by e-mail at G.Smith@optometry.unimelb.edu.au for copies of the programs referred to in this paper.

13. This does not necessarily mean that the coma error is greater than the spherical aberration. The coma coefficient gives the error at the edge of a 1-mm-radius pupil. If the pupil radius is \( \rho \), the level of coma error at the edge of the pupil is the value of the coma coefficient multiplied by \( \rho^3 \), while the level of spherical aberration is the value of the spherical aberration coefficient multiplied by \( \rho^4 \).